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Singular value decomposition of the velocity-reflector depth tradeoff, Part 2: High-resolution analysis of a generic model

Christof Stork*

ABSTRACT

The symmetries of a block circulant matrix significantly reduce the computational expense of the singular value decomposition (SVD) of the variable velocity inverse problem for a generic reflection seismology model. As a result, the decomposition does not suffer from edge effects or parameterization artifacts that are associated with small model spaces. Using this approach, we study the eigenvector and eigenvalue characteristics for a generic model of a size as large as is used with a variety of iterative inversion techniques (>100 000 parameters).

Singular value decomposition of the raypath inverse problem of a discretized generic seismic model having one reflector indicates that the eigenvalue distribution for the inverse problem is nonuniform, with a large concentration near 0 and a gap near 0.4. All but the long horizontal wavelength reflector-depth variations cannot be uniquely resolved from velocity variations. Lateral velocity variations serve to significantly reduce the ability of seismic data to resolve reflector depth for most of the horizontal wavelength components shorter than twice the cable length. As a result, automatic velocity analysis methods may not be able to resolve reflector variations when the velocity field is allowed to take on an arbitrary structure.

INTRODUCTION

A main objective of a seismic reflection survey is to accurately map the depth structure of one or more reflectors. However, with the limited angular ray coverage available in a reflection survey, a potential ambiguity exists between reflector depth and the overlying velocity. Apparent reflector structure may, in some cases, be an artifact caused by unresolved velocity variations above the reflector. A task of some velocity analysis methods is to resolve the velocity variations from false reflector depth variations.

Numerous methods have been proposed for using seismic data to resolve velocity in the presence of lateral variations (Bishop et al., 1985; Tarantola, 1986; Bording et al., 1987; Mora, 1987; Kennett et al., 1988; Stork and Clayton, 1991; Williamson, 1986; Sword, 1987; Fowler, 1988; van Trier, 1990; Biondi, 1990; Etgen, 1990; van der Made, 1988; Sherwood et al., 1986; Julien et al., 1988). These methods differ considerably, but many share the characteristic of resolving the reflector structure and the velocity variations using the prestack energy transmitted through the medium having unknown velocity and reflected off of an underlying interface. The relationship between data and model is determined by the path of energy propagation.

Most of these velocity analysis methods involve the inversion of very large linear systems that require efficient iterative techniques. The size of these linear systems makes the analysis of their resolution characteristics difficult. Frequently, the prime tool for analyzing resolution characteristics is repeated test inversions of synthetic models.

The method of singular value decomposition (SVD) (Lanczos, 1961) determines the eigenvalues and eigenvectors of a linear system that allows the resolution characteristics of the system to be analyzed analytically. The eigenvalues give a measure of the resolution of the corresponding eigenvector. By identifying a group of eigenvectors corresponding to reflector depth variations, one can determine the approximate resolution of reflector depths and the inversion parameters necessary for depth resolution. Of particular interest, is how well the reflector is resolved in the presence of velocity variations.

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*Formerly Stanford Exploration Project, Stanford University, Department of Geophysics, Stanford, CA; presently Advance Geophysical Co., 7409 S. Alton Ct., Suite 100, Englewood, CO 80112.
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Moreover, the eigenvalue distribution is important for predicting the performance of iterative inversion techniques.

Conventional methods for producing an SVD (Golub and Reinsch, 1970) are significantly more computationally expensive than iterative inversion techniques. Because of this additional expense, several less costly procedures are used here to produce results with resolution as high as those from the iterative techniques. Taking advantage of the block circulant and symmetric nature of the matrix resulting from a generic model significantly increases the model size that can be computed with SVD. I parameterize the velocity field as gradients between the points instead of using constant velocity cells. Edge effects resulting from the roll-in or roll-out are avoided by using a wide survey that is wrapped around, such that the rays exiting one side of the model enter on the other side.

I perform a high-resolution SVD of the traveltime inversion problem used in tomographic velocity analysis of seismic reflection data (Bishop et al, 1985; Bording et al., 1987; Williamson, 1986; Kennett et al., 1988; Stork and Clayton, 1991). The geometry of the inverse problem is presented schematically in Figure 1. The basic model consists of a velocity field, parameterized by a two-dimensional (2-D) rectangular grid of points (not cells), and one reflector, parameterized by a regularly spaced series of points.

Bube and Resnick (1984), Bube et al. (1985), and Bishop et al. (1985) performed an SVD analysis with similar goals as those in this study. Their analysis was limited by the computational expense required for SVD. As a result, they used relatively large cell sizes and a narrow model width, which the experience of Stork (1988a) indicates corrupts the results. Bube et al. (1989) avoid the digitization artifacts by analyzing the problem using continuous functions of arbitrary resolution. They show that all aspects of reflector depth are theoretically resolvable with perfect data. Bickel (1990) and Toldi (1985) address the resolution of velocity variations from reflector depth using stacking velocities. The results of these studies indicate that laterally variable velocity severely degrades reflector depth resolution.

The SVD results indicate that even for a large range of ray angles, only the long spatial wavelengths of reflector depth can be resolved with confidence from certain patterns of velocity variations. Some reflector components correspond to eigenvalues as low as 0.01 and cannot be effectively resolved. The eigenvalue distribution is very nonuniform with a large density near 0, which affects iterative inversion methods.

THE FORWARD PROBLEM

Traveltime deviations from the discretized velocity field variations of the reference model is expressed using constant velocity cells as:

$$\Delta t_k = \sum_j \ell_{kj} \cdot \Delta s_j \qquad (1)$$

where:

Δt_k = the traveltime deviation of the kth ray.
ℓ_{kj} = the path length of the kth ray in the jth cell.
Δs_j = the slowness variation of the jth cell. (Slowness is the inverse of velocity.)

This model treats the effect of raypath perturbations as second order and therefore negligible.

In matrix notation equation (1) is

$$\mathbf{\Delta t} = \underset{\sim}{\mathbf{L}} \Delta \underset{\sim}{\mathrm{s}}, \qquad (1a)$$

where:

$\mathbf{\Delta t}$ = vector of traveltime deviation predicted from reference model.
$\underset{\sim}{\mathbf{L}}$ = matrix of path lengths of the rays in each cell.
$\Delta \mathrm{s}$ = vector of slowness deviations from reference model.

The effect of reflector variations on traveltimes is included in this formulation as in Part 1 of this paper (Stork, 1992).

In this implementation, the velocity field is parameterized as a continuous field rather than a blocky one with cells. The field is specified at regularly spaced points in a rectangular grid. Between the points, the field is defined as a linear interpolation of the values at the nearby points, as shown in Figure 2. The effect of the points on the traveltime of a given ray (analogous to the path length of the ray in a cell) is a linear interpolation of the points near the raypath.

I will analyze a variation of the forward problem with weights applied to data and model spaces:

$$\underset{\sim}{\mathbf{D}}^{1/2}\mathbf{\Delta t} = (\underset{\sim}{\mathbf{D}}^{1/2}\underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{S}}^{1/2})\underset{\sim}{\mathbf{S}}^{-1/2}\Delta \mathrm{s}, \qquad (1b)$$

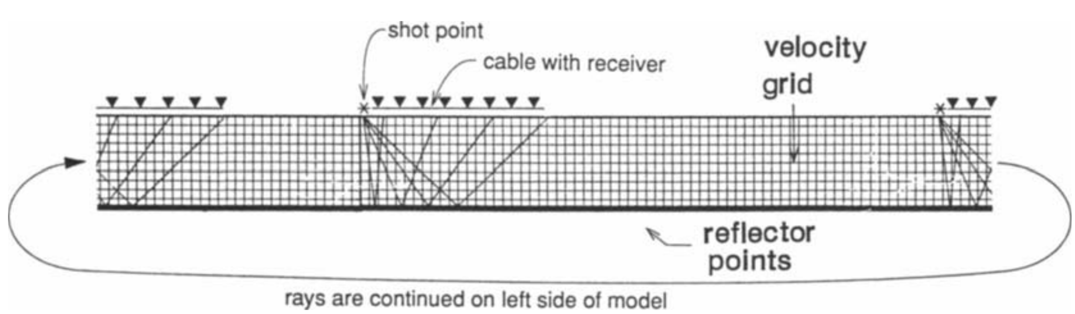

FIG. 1. Diagram for the generic model used in the SVD analysis. The actual model consists of 8192 × 40 points. The reflector is at the bottom of the model. Ray paths are straight. Rays exiting one side of the model are wrapped around to the other side creating a circular reflection survey. The velocity is defined at the corner of the cells with the velocity in the cell a linear interpolation of the surrounding points (see Figure 2).

where $\underset{\sim}{D}$ and $\underset{\sim}{S}$ are diagonal matrices with elements:

$\underset{\sim}{D}_{kk} = \sum_i \ell_{ki}$, which is the path length of the ray in the cell.

$\underset{\sim}{S}_{ii} = \sum_k \ell_{ki} + \varepsilon$, which is the sum of the ray segment lengths in the ith cell plus a damping factor to avoid a zero divide.

Equation (1b) is rewritten

$$\mathbf{d} = \underset{\sim}{\mathbf{A}}\mathbf{m}, \tag{1c}$$

where

$\mathbf{d} = \underset{\sim}{\mathbf{D}}^{1/2}\Delta\mathbf{t}$,
$\underset{\sim}{\mathbf{A}} = (\underset{\sim}{\mathbf{D}}^{1/2}\underset{\sim}{\mathbf{L}}\underset{\sim}{\mathbf{S}}^{1/2})$,
$\mathbf{m} = \underset{\sim}{\mathbf{S}}^{-1/2}\Delta\mathbf{s}$.

The square root of a diagonal matrix is defined here as the matrix of the square root of its elements.

I analyze the eigenvectors and eigenvalues of matrix A. These weights result from the use of the Dines and Lytle (1979) back projection formula. I use these weights because they set the maximum eigenvalue of A to 1.0 and provide for a uniform null space in the presence of a nonuniform ray illumination [Comer and Clayton (1985, unpublished manuscript); Stork and Clayton, 1991]. The weights are not expected to significantly affect the SVD results. Since data coverage is uniform throughout the model, model weighting is uniform. Data weighting is mild with the rays at an angle of 45 degrees weighted only 12 percent less than vertical rays.

Since I address only the model space eigenvectors $\underset{\sim}{\mathbf{V}}$, the SVD analysis is performed on:

$$\underset{\sim}{\mathbf{A}}^T\underset{\sim}{\mathbf{A}} = (\underset{\sim}{\mathbf{U}}\underset{\sim}{\Sigma}\underset{\sim}{\mathbf{V}}^T)^T(\underset{\sim}{\mathbf{U}}\underset{\sim}{\Sigma}\underset{\sim}{\mathbf{V}}^T) = \underset{\sim}{\mathbf{V}}\underset{\sim}{\Sigma}^2\underset{\sim}{\mathbf{V}}^T, \tag{1d}$$

where

$\underset{\sim}{\mathbf{V}}$ = an orthonormal matrix of model space eigenvectors.
$\underset{\sim}{\mathbf{U}}$ = an orthonormal matrix of data space eigenvectors.
$\underset{\sim}{\Sigma}$ = a diagonal matrix of the eigenvalues.

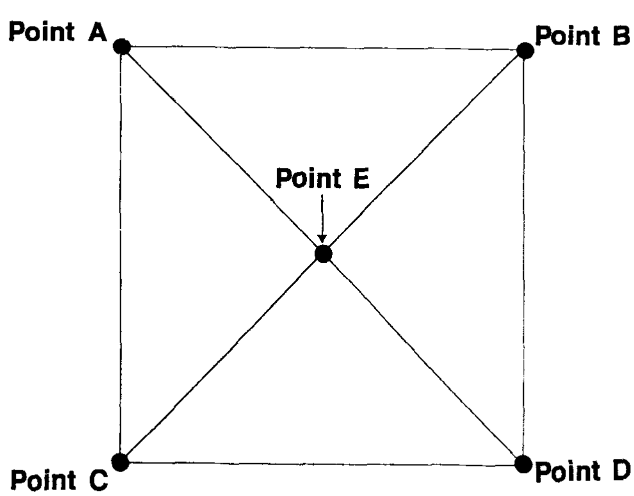

FIG. 2. The velocity is interpolated in a cell based on the four points (points A-D) at the corners of a cell. The velocity within a triangle is a linear gradient of the three points at its corners. The velocity of point E is the average of the four other points.

Not computing data space eigenvectors allows for the use of a very large data space that is essentially a continuous field that produces minimal sampling artifacts. Using this formulation, however, has the drawback of computing the square of the eigenvalues, presenting potential numerical problems for small eigenvalues.

While the computation of $\underset{\sim}{\mathbf{A}}^T\underset{\sim}{\mathbf{A}}$ is generally an expensive procedure comparable to the SVD computation itself, the sparseness of the matrix makes the computation relatively inexpensive.

The SVD procedure is generally so expensive that computers are not expected to be powerful enough in the near future for the method to be applied on the size of model spaces used with iterative techniques on data applications. [Experience (Stork, 1988a) indicates 10^5 CRAY-2 CPU hours are necessary for a matrix of a size 100 000 × 100 000 using a SVD routine optimized for the machine!] By taking advantage of the symmetries of a generic reflection survey, SVD can be formulated in the horizontal wavenumber domain to significantly increase the model size that can be analyzed.

SVD OF A BLOCK CIRCULANT MATRIX

SVD is performed on a generic 2-D model shown in Figure 1. To avoid effects of the limited ray coverage at the edges of the survey, raypaths exiting at one side of the model are entered on the other side, producing a continuous, circular reflection survey with complete ray coverage everywhere. This approach avoids the edge effects. Consistent results from models run with different circumferences indicate the model was wide enough (aspect ratio of 40:1) so that the results are not affected by the wraparound.

The matrix for SVD produced by our circular geometry of the generic model is block circulant and symmetric. A plain circulant matrix consists of identical rows except that each is shifted over so the elements along a diagonal are identical. The elements from one side are wrapped around to the other side. This implies

$$\underset{\sim}{C}^{\text{plain}}_{i,j} = \underset{\sim}{C}^{\text{plain}}_{i+k,j+k}, \quad \text{and} \quad \underset{\sim}{C}^{\text{plain}}_{N,j} = \underset{\sim}{C}^{\text{plain}}_{0,j+1}. \tag{2}$$

For a block circulant matrix, this pattern is repeated only every Mth row or column. This implies

$$\underset{\sim}{C}^{\text{block}}_{i,j} = \underset{\sim}{C}^{\text{block}}_{i+(k\times M),j+(k\times M)}, \quad \text{and}$$

$$\underset{\sim}{C}^{\text{block}}_{N-M+\ell,j} = \underset{\sim}{C}^{\text{block}}_{\ell,j+M}, \tag{2a}$$

where M is the dimension of the blocks. Uniform ray coverage for the generic model used here produces a block circulant matrix because locations at the same depth within every model are identical. The dimension of the blocks M is the number of cells in a column of the model.

Our matrix $\underset{\sim}{\mathbf{A}}^T\underset{\sim}{\mathbf{A}}$ is also symmetric. A plain circulant and symmetric matrix can be represented using the Fourier cosine series:

$$\underset{\sim}{C}^{\text{plain}}_{i,j} = \frac{2}{N} \cdot \sum_{k=1}^{N/2} \lambda_k \cdot \cos\left(\frac{2\cdot\pi}{N} \cdot k \cdot (i-j)\right). \tag{2b}$$

Only the cosines are needed because the matrix is symmetric. Using

$$\cos(i-j) = \cos(i)\cdot\cos(j) + \sin(i)\cdot\sin(j) \quad (2c)$$

the expression can be rewritten:

$$\underset{\sim}{C}^{\text{plain}}_{i,j} = \frac{2}{N}\cdot\sum_{k=1}^{N/2}\lambda_k\cdot\left(\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot i\right)\cdot\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot j\right) + \sin\left(\frac{2\cdot\pi}{N}\cdot k\cdot i\right)\cdot\sin\left(\frac{2\cdot\pi}{N}\cdot k\cdot j\right)\right). \quad (2d)$$

Since the harmonic functions are orthogonal, this summation can be identified with the singular value decomposition

$$\underset{\sim}{C}^{\text{plain}} = \underset{\sim}{V}\underset{\sim}{\Sigma}\underset{\sim}{V}^T, \quad (2e)$$

where the elements of $\underset{\sim}{V}$ are:

$$\underset{\sim}{V}_{i,j} = \frac{2}{N}\cos\left(\frac{2\cdot\pi}{N}\cdot\frac{(i-1)}{2}\cdot j\right) \quad \text{for odd } i$$

$$= \frac{2}{N}\sin\left(\frac{2\cdot\pi}{N}\cdot\frac{i}{2}\cdot j\right) \quad \text{for even } i,$$

and the elements of $\underset{\sim}{\Sigma}$ are:

$$\underset{\sim}{\Sigma}_{2i,2i} = \underset{\sim}{\Sigma}_{2i+1,2i+1} = \lambda_i.$$

Thus, performing an SVD on a circulant and symmetric matrix merely performs a Fourier decomposition of the invariant row of the matrix. The eigenvectors are the sinusoids and the eigenvalues are the Fourier spectrum. Performing a Fourier transform on a row is vastly cheaper than performing an SVD on the whole matrix.

To consider a block circulant and symmetric matix, I represent it as a plain circulant matrix with multidimensional elements. Each multidimensional element, $\underset{\sim}{C}^{\text{block}}_{i,j}$, can be considered to be a submatrix of size $M \times M$. The elements of each submatrix are represented as $(\underset{\sim}{C}^{\text{block}}_{i,j})_{k,\ell}$. The i and j indices refer to the position of a block while the k and ℓ indices refer to the position of an element within a block.

The multidimensional eigenvectors and eigenvalues analogous to the previous ones are written:

$$\underset{\sim}{V}_{i,j} = \mathbf{v}_i\,\frac{2}{N}\cos\left(\frac{2\cdot\pi}{N}\cdot\frac{(i-1)}{2}\cdot j\right) \quad \text{for odd } i$$

$$= \mathbf{v}_i\,\frac{2}{N}\sin\left(\frac{2\cdot\pi}{N}\cdot\frac{i}{2}\cdot j\right) \quad \text{for even } i,$$

and the elements of $\underset{\sim}{\Sigma}$ are:

$$\underset{\sim}{\Sigma}_{2i,2i} = \underset{\sim}{\Sigma}_{2i+1,2i+1} = \underset{\sim}{\boldsymbol{\lambda}}_i,$$

where $\underset{\sim}{\mathbf{v}}_i$ and $\underset{\sim}{\boldsymbol{\lambda}}_i$ are themselves matrices. The singular value decomposition $\underset{\sim}{C}^{\text{block}} = \underset{\sim}{U}\underset{\sim}{\Sigma}\underset{\sim}{U}^T$ holds when matrix $\underset{\sim}{\boldsymbol{\lambda}}_i$ is diagonal and matrix $\underset{\sim}{\mathbf{v}}_i$ is orthonormal. The singular value decomposition can be rewritten in terms of the Fourier series:

$$\underset{\sim}{C}^{\text{block}}_{i,j} = \frac{2}{N}\cdot\sum_{k=1}^{N/2}\left(\underset{\sim}{\mathbf{v}}_k\underset{\sim}{\boldsymbol{\lambda}}_k\underset{\sim}{\mathbf{v}}_k^T\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot i\right)\cdot\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot j\right) + \underset{\sim}{\mathbf{v}}_k\underset{\sim}{\boldsymbol{\lambda}}_k\underset{\sim}{\mathbf{v}}_k^T\sin\left(\frac{2\cdot\pi}{N}\cdot k\cdot i\right)\cdot\sin\left(\frac{2\cdot\pi}{N}\cdot k\cdot j\right)\right), \quad (2f)$$

or

$$\underset{\sim}{C}^{\text{block}}_{i,j} = \frac{2}{N}\cdot\sum_{k=1}^{N/2}\underset{\sim}{\mathbf{v}}_k\underset{\sim}{\boldsymbol{\lambda}}_k\underset{\sim}{\mathbf{v}}_k^T\cdot\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot(i-j)\right). \quad (2g)$$

Performing an inverse Fourier transform for $j = 0$ produces:

$$\frac{2}{N}\cdot\sum_{i=1}^{N/2}\underset{\sim}{C}^{\text{block}}_{(i,0)}\cdot\cos\left(\frac{2\cdot\pi}{N}\cdot k\cdot i\right) = \underset{\sim}{\mathbf{v}}_k\underset{\sim}{\boldsymbol{\lambda}}_k\underset{\sim}{\mathbf{v}}_k^T. \quad (2h)$$

Thus the SVD of the block circulant and symmetric matrix has been reduced to performing Fourier transforms followed by numerous small SVD's of the block for each horizontal wavenumber k. It is possible to perform an SVD of a 300 000 $\times$ 300 000 matrix with block size 40 on a conventional workstation (Sparc Station 1) within an hour.

THE GENERIC MODEL

The model used for the SVD has 8192 points in the horizontal direction and 40 points in the vertical direction for the velocity field and 8192 points for the reflector. The resulting model space is (41 $\times$ 8192) 335 872. The model aspect ratio is 20:1. The background velocity is constant and the reflector is flat producing straight raypaths. The cable length is twice the depth to the reflector giving a maximum ray angle of 45 degrees. Reflector weight (the w factor of part 1 of this paper) is 0.5. Rays exiting one side of the model are wrapped around to the other side. Figure 1 depicts the ray coverage. A very dense reflection survey (4 shots per cell width, 8 receivers per cell width) is used to compute the $\underset{\sim}{A}^T\underset{\sim}{A}$ matrix such that ray coverage can be considered to be essentially continuous.

SVD OF THE GENERIC MODEL IN THE WAVENUMBER DOMAIN

The eigenvalues of this model are plotted in descending order in Figure 3. As predicted from Comer and Clayton (1985, unpublished manuscript), all eigenvalues are between 1.0 and 0.0. The distribution of the eigenvalues, plotted in Figure 4, is uneven. The greatest density of eigenvalues is near 0. A gap where no eigenvalues exist occurs near 0.5.

Analysis of the eigenvectors indicates they can be separated into three approximate groups. The first group (group A), an example of which is plotted in Figure 5a, represents the constructive interference of velocity and reflector depth. This group corresponds to the eigenvector (1.0, 1.0) of the two-parameter model in part 1 of this paper. The second group (group B), examples of which are plotted in Figures 5b and c, has a small reflector component. This group represents velocity variations that are not coupled with reflector variations. The third group (group C), examples of which are plotted in Figures 5d and e, represent the destructive interference of velocity and reflector depth. This group corresponds to the eigenvector (1.0, −1.0) of the two-parameter model in part 1 of this paper.

Group A has uniformly large eigenvalues. All eigenvalues larger than 0.5 correspond to this group. Group B can have large and small eigenvalues. Eigenvectors in this second group with large eigenvalues (0.1 − 0.4) will generally have velocity variations that are higher than they are wide (vertically oriented), such as Figure 5b. These eigenvectors with small eigenvalues (0.0 − 0.1) will generally have velocity variations that are wider than they are high (horizontally oriented), such as in Figure 5c. Group C generally has more eigenvalues varying from 0.1 to 0.0.

Since there are several thousand eigenvectors of interest in this model, analysis of the eigenvectors is performed using plots of the eigenvector characteristics in Figures 6–11.

The eigenvectors consist of just one horizontal wavenumber component because of the block circulant nature of the matrix used for SVD. Figure 6 is a plot of the horizontal

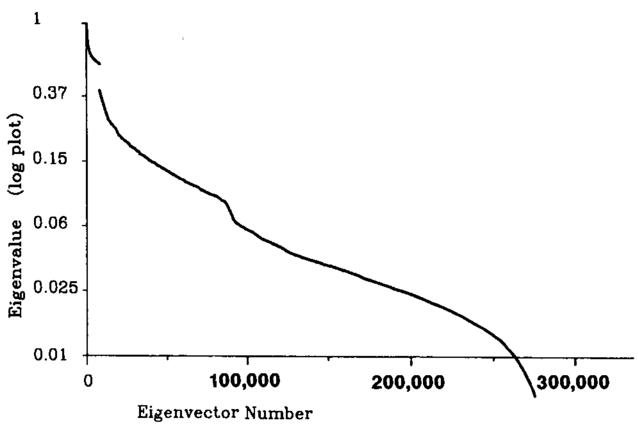

FIG. 3. Eigenvalues distribution plotted in descending order on a log scale. There are 335 872 eigenvalues. Note that a gap appears near eigenvalue 0.5.

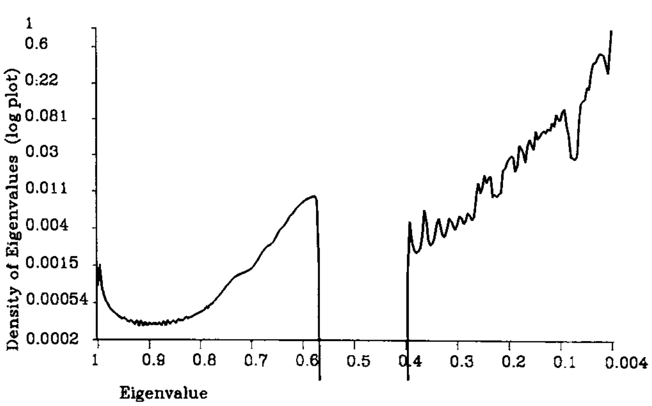

FIG. 4. Distribution of the eigenvalues plotted on a log scale. The left scale is normalized so that the maximum value is 1.0. The distribution is very nonuniform, which affects some iterative matrix inverse methods.

wavenumber of an eigenvector versus its eigenvalue. A wavenumber of 1.0 corresponds to a wavelength equal to the depth of the reflector. Below a wavenumber of 0.5, few eigenvectors have eigenvalues above 0.01. At wavenumber 0.5, the significant change is probably related to the cable length being twice the depth of the reflector.

There are several eigenvector trends in this plot that form lines. The lines appear to become complicated below an eigenvalue of 0.06, yet are still discernable. Tests of whether the eigenvalues less than 0.06 are numerical artifacts were performed by repeated SVD analysis models with different features, such as cell size, cell shape, and different parameters of the model space. The consistent results do not indicate any numerical artifacts.

To analyze the resolution of the reflector depth, the fraction of the eigenvectors' components (in terms of rms) that lies in the reflector portion of the model is plotted versus the eigenvalue of the eigenvector in Figure 7. The points above eigenvalue 0.5 are group A that correspond to the constructive interference of velocity and reflector depth. Those reflector components in eigenvectors with very small eigenvalues cannot be resolved from the corresponding velocity pattern in the eigenvector. The points below eigenvalue 0.30 with a significant fraction of reflector components are group C that correspond to the destructive interference of velocity and reflector depth. Two clumps of eigenvectors with high reflector components exist near eigenvalue 0.1 and 0.015. Several eigenvectors with significant reflector components have eigenvalues below 0.01. There are also a significant number of eigenvectors with very small reflector components. These eigenvectors are a mix of groups B and C.

Figure 8a plots the eigenvalues versus the horizontal wavenumber of the eigenvectors without a significant fraction of reflector components (<0.02). These eigenvectors are group B. This group contains most eigenvectors in the middle range of the spectrum, between 0.50 and 0.10, as well as many below 0.10. Figure 8b plots those eigenvectors with a significant fraction of reflector component (<0.02). These eigenvectors consist of groups A and C. Group A is marked on the figure. For the rest of the eigenvectors, a fairly complicated pattern is seen, with eigenvalues ranging from 0.3 to 0.01 with most below 0.10.

To further characterize the resolution of the reflector for the eigenvectors in group C, Figure 9 shows the eigenvalue which needs to be inverted to resolve 90 percent of the reflector components of group C for a given wavenumber. Figures 10 and 11 plot the eigenvalue necessary for a 70 and 50 percent inversion, respectively. For horizontal wavenumbers near 0, the eigenvectors of group C have an eigenvalue near 0.1. This result is consistent with the results of the two-parameter model of Part 1 of this paper, where the eigenvalue of the two-parameter eigenvector (1.0, −1.0) is also 0.1.

For eigenvectors with horizontal wavenumbers in the range of 0.1 − 0.4, the eigenvalue drops significantly, to less than 0.01 at a horizontal wavenumber of 0.3. At this wavenumber, the velocity variations are able to mimic the hyperbolic moveout of the reflector depth variations.

For eigenvectors with horizontal wavenumbers of 0.5 and greater, resolution of the reflector components generally increases and is variable. Notches exist where nearby wave-

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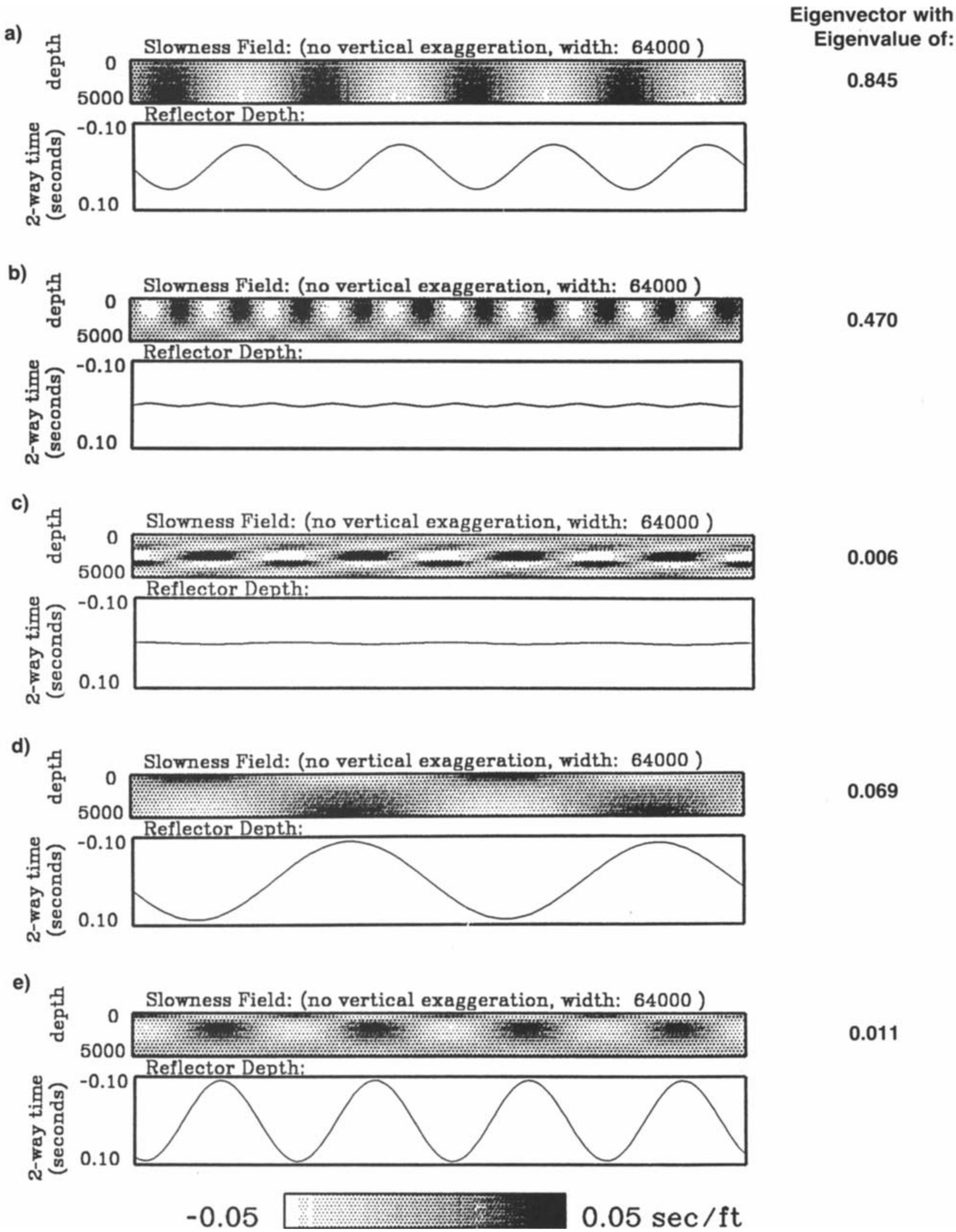


FIG. 5. Sample eigenvectors for the generic model. (a) This eigenvector with an eigenvalue of 0.845 represents group A, which is the constructive interference of the velocity and reflector depth. This group, which consists of all eigenvalues above 0.5, adjusts the model to match the average two way traveltime in the data. The eigenvector (1.0, 1.0) of the two-parameter model is an example of this group. Since the function of adjusting the model to match the average two-way traveltime is generally performed by other methods in seismic processing and interpretation, this group is not a major objective of velocity analysis methods. (b) Eigenvalue: 0.470, and (c) Eigenvalue: 0.006. These eigenvectors represent group B, which contain little reflector components. The velocity variations of this group do not significantly effect the reflector depth. Some eigenvectors of this group are large, such as that of (b), while others are small, such as (c). Since a main objective of velocity analysis is reflector depth, this group is not of prime significance. (d) Eigenvalue: 0.069 and (e) Eigenvalue: 0.011. These eigenvectors represent group C, which is the destructive interference of velocity and reflector depth. This group corresponds to the eigenvector (1.0, −1.0) of the two-parameter model. This group has eigenvalues ranging from 0.1 to less than 0.01. The resolution of these eigenvectors is a major objective of velocity analysis, yet it appears that some of them are too small to resolve.

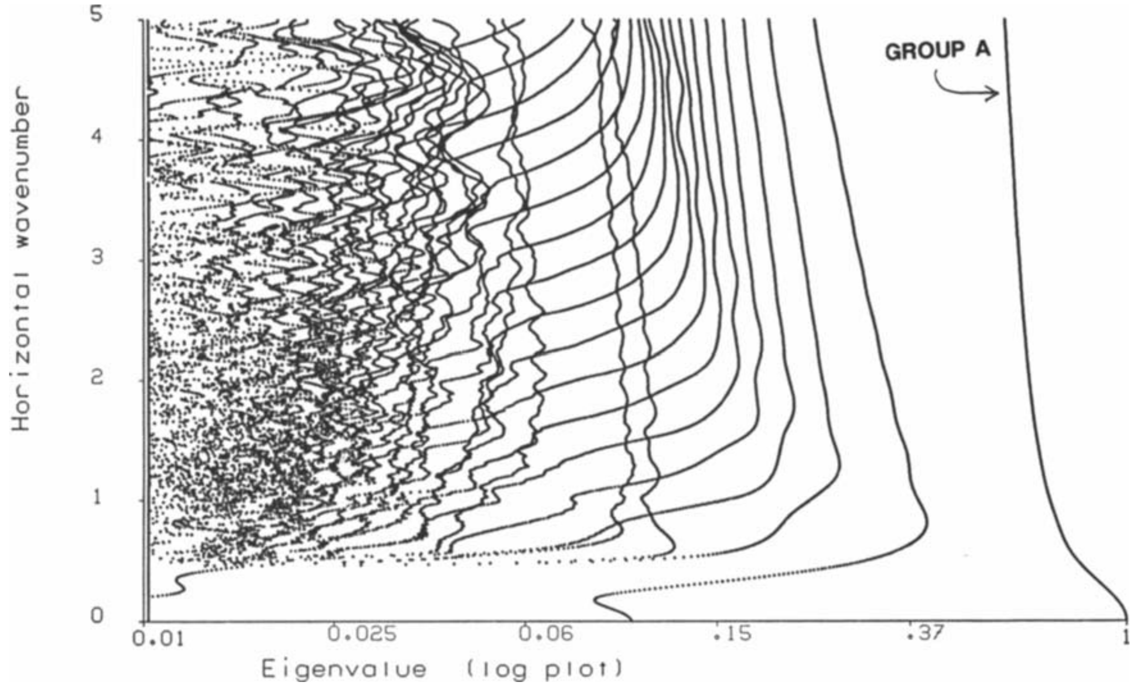


FIG. 6. The horizontal wavenumber of an eigenvector verses its eigenvalue characteristic. A horizontal wavenumber of 1.0 represents the same wavelength as the depth to the reflector. The eigenvectors fall into many distinct trends. The sharp jump at a wavenumber of 0.5 is probably related to the cable length having a length equal to twice the depth of the reflector. Group A is trend marked on the far right. The rest of the eigenvectors consist of both group B and C.

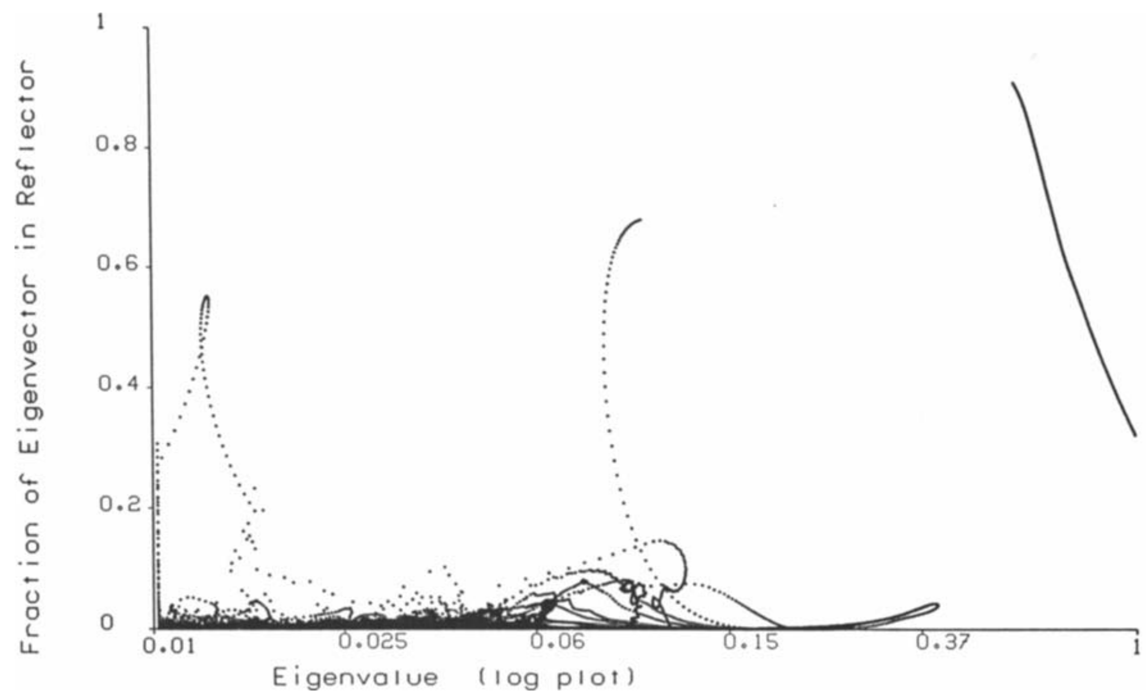
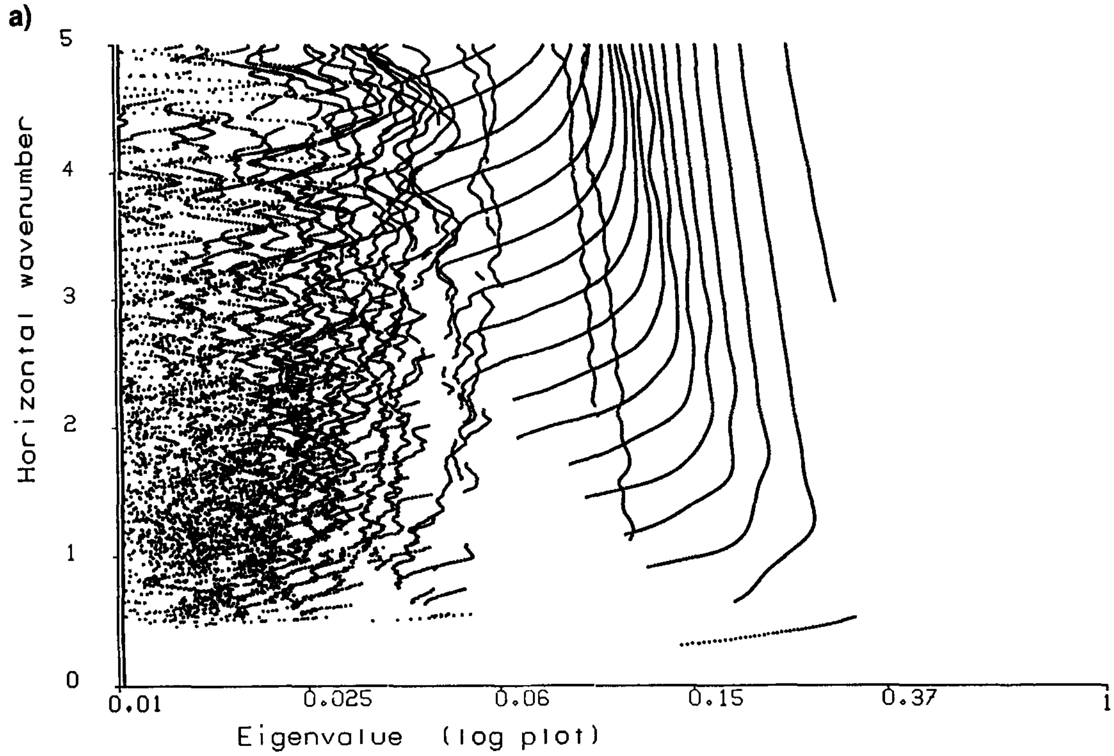


FIG. 7. The fraction of an eigenvector (in terms of rms) that lies in the reflector component versus its eigenvalue. The points above eigenvalue 0.5 are group A that correspond to the constructive interference of velocity and reflector depth. The points with a significant fraction of reflector components below an eigenvalue of 0.10 are group C that correspond to the destructive interference of velocity and reflector depth. Two clumps of eigenvectors with high reflector component exist near 0.1 and 0.015. Several eigenvectors with significant reflector components have eigenvalues below 0.01. Those reflector components in eigenvectors with very small eigenvalues ($<\sim 0.03$) cannot be resolved from the corresponding velocity pattern in the eigenvector. There are also a significant number of eigenvectors with very small reflector components. These eigenvectors are a mix of groups B and C.



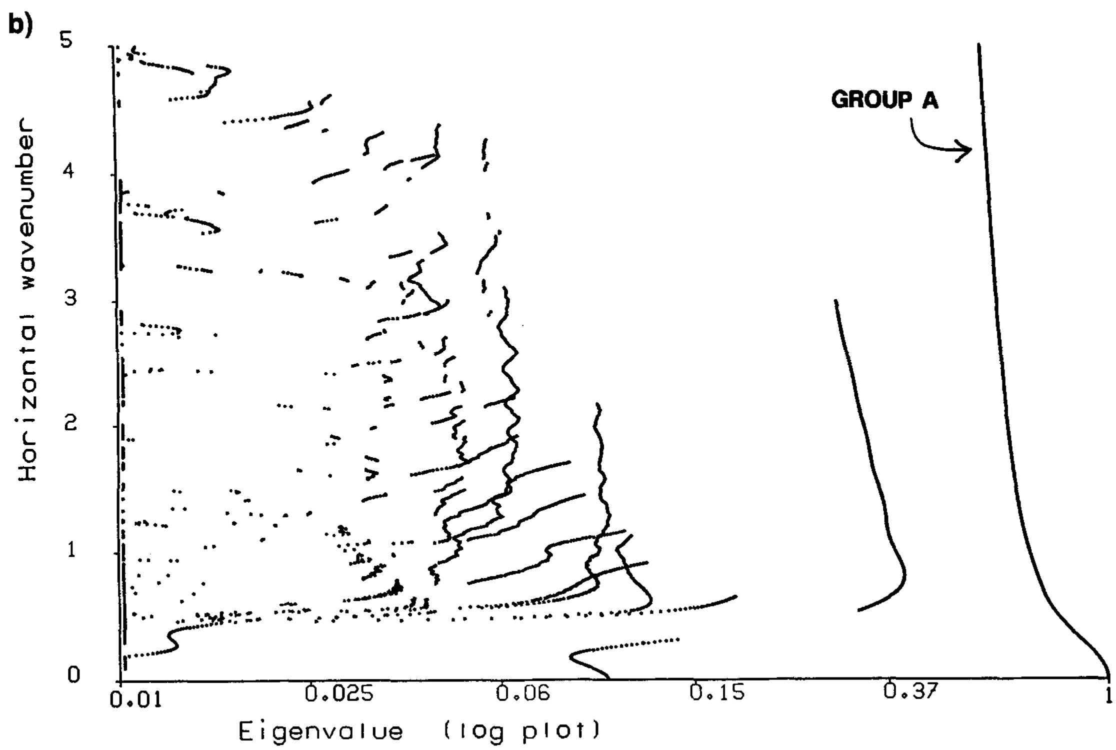


FIG. 8. (a) Eigenvalues versus the horizontal wavenumber of the eigenvectors without a significant fraction of reflector components. These eigenvectors are group B. This group contains most eigenvectors in the middle range of the spectrum, between 0.50 and 0.10, as well as many below 0.10. (b) Eigenvectors with a significant fraction of reflector component. These eigenvectors consist of group A and C. Group A is marked on the figure. For the rest of the eigenvectors, we see a fairly complicated pattern with eigenvalues ranging from 0.3 to 0.01, but most are below 0.10.

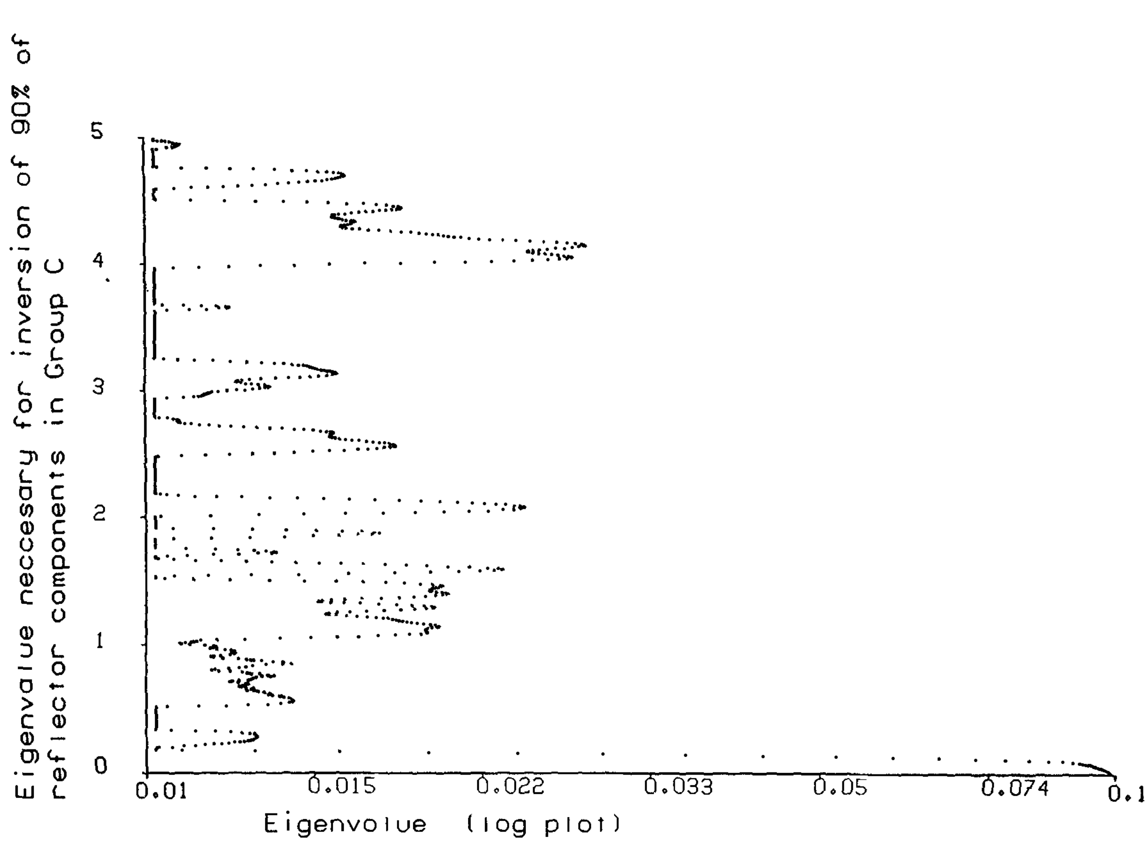


FIG. 9. Eigenvalue which needs to be inverted to resolve 90 percent of the reflector components of group C for a given wavenumber. A small eigenvalue means that the wavenumber is poorly resolved.

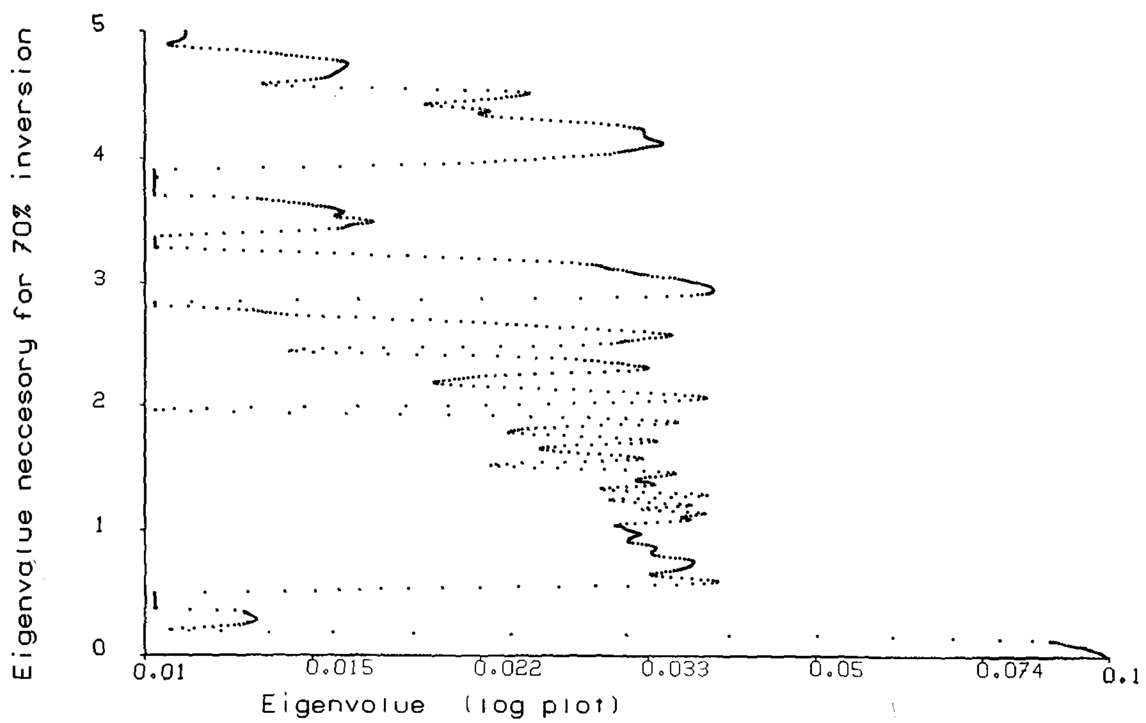


FIG. 10. Eigenvalue which needs to be inverted to resolve 70 percent of the reflector components of group C for a given wavenumber.

numbers have significantly different resolution. Some reflector components have very small eigenvalues while others have eigenvalues in the range of 0.10.

DISCUSSION

The relationship between eigenvectors and eigenvalues to structures of interest is valuable knowledge for the application of velocity analysis methods that address lateral variations (Stork, 1988a). For instance, the optimal range of eigenvalues over which to perform an inversion should be just wide enough to include the desired objective, yet as narrow as possible to reduce the effects of noise and keep computer usage as low as possible. In some cases, the eigenvalue associated with the structure will be too small and cannot be effectively inverted. The inability to resolve certain structural features should be known, so the results of the inversion are cautiously interpreted.

Moreover, knowledge of the distribution of the eigenvalues can help predict the performance of iterative acceleration techniques. For instance, the very uneven distribution of the eigenvalues may significantly affect the conjugate gradient algorithm (Stork, 1988b).

A main goal of this SVD analysis of the surface reflection seismology experiment is to characterize the ambiguity between velocity and reflector depth in the presence of velocity variations. This ambiguity is represented by the eigenvectors of Group C. The resolution of various wavenumbers of Group C, highlighted in Figures 9 and 10, varies considerably. Some components are very poorly resolved with eigenvalues near 0.01. This poor resolution indicates that certain patterns of velocity variations can produce nearly the same traveltime patterns as some reflector depth perturbations. In particular, Figures 9–11 show that resolution is poor for the normalized wavenumber 0.5, which corresponds to twice the cable length. Bube et al. (1989), however, demonstrate that all reflector components are indeed resolvable to some degree and thus the eigenvalues never actually reach zero.

We anticipate these results for the traveltime inverse problem apply to velocity analysis methods with different data formulations as well. For instance, Santosa and Symes (1989) come to a similar conclusion using waveform inversion for one reflector. Their results indicate, however, that additional reflectors improve the overall resolution of the model.

CONCLUSION

In some cases, velocity variations cannot be resolved effectively from reflector variations using traveltime data from one reflector. Velocity analysis methods that use different forms of data may also share this problem. Consequentially, fully automated velocity analysis methods that address lateral variations may not be robust if the velocity is allowed to vary without restriction.

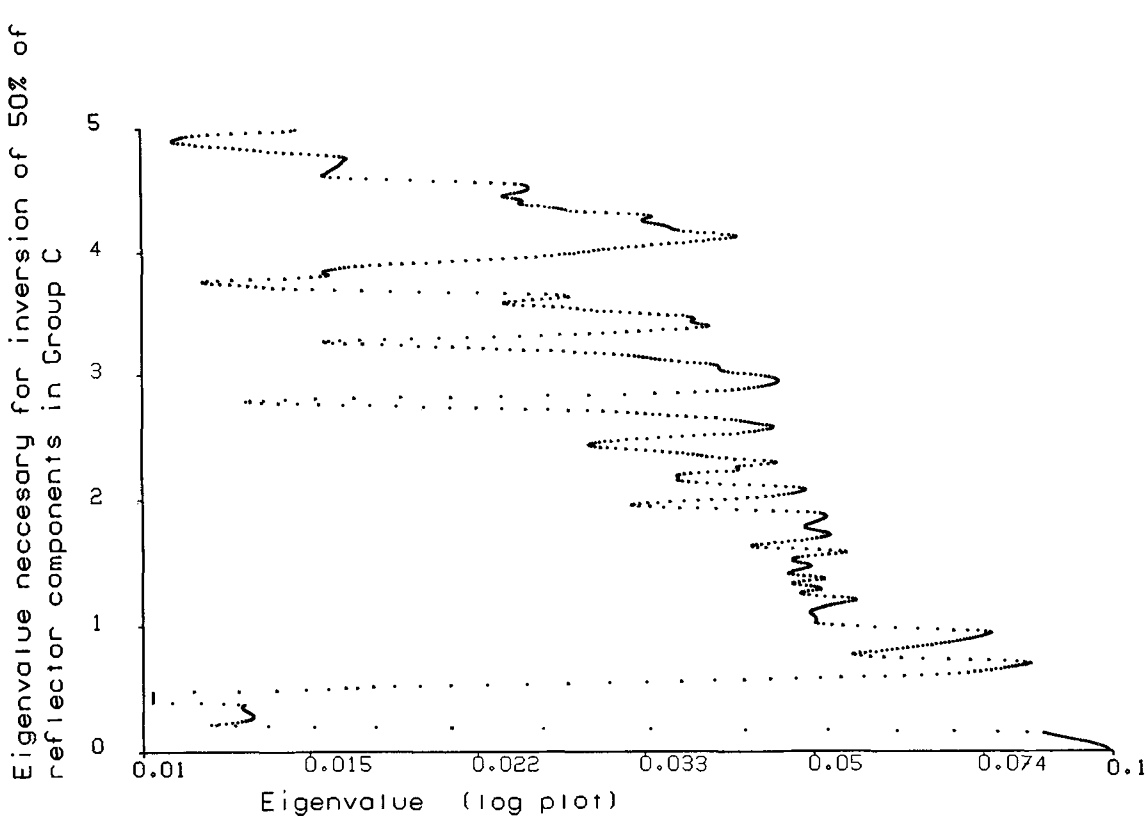


FIG. 11. Eigenvalue which needs to be inverted to resolve 50 percent of the reflector components of group C for a given wavenumber.

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